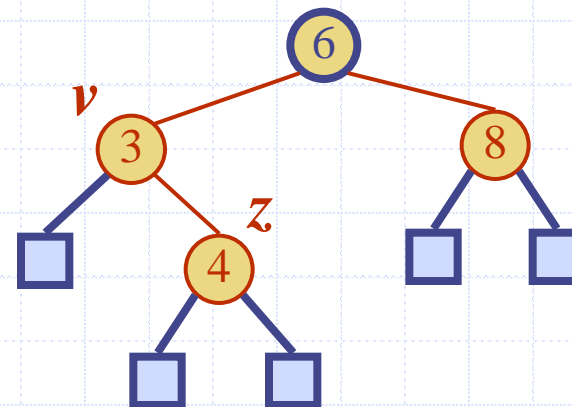
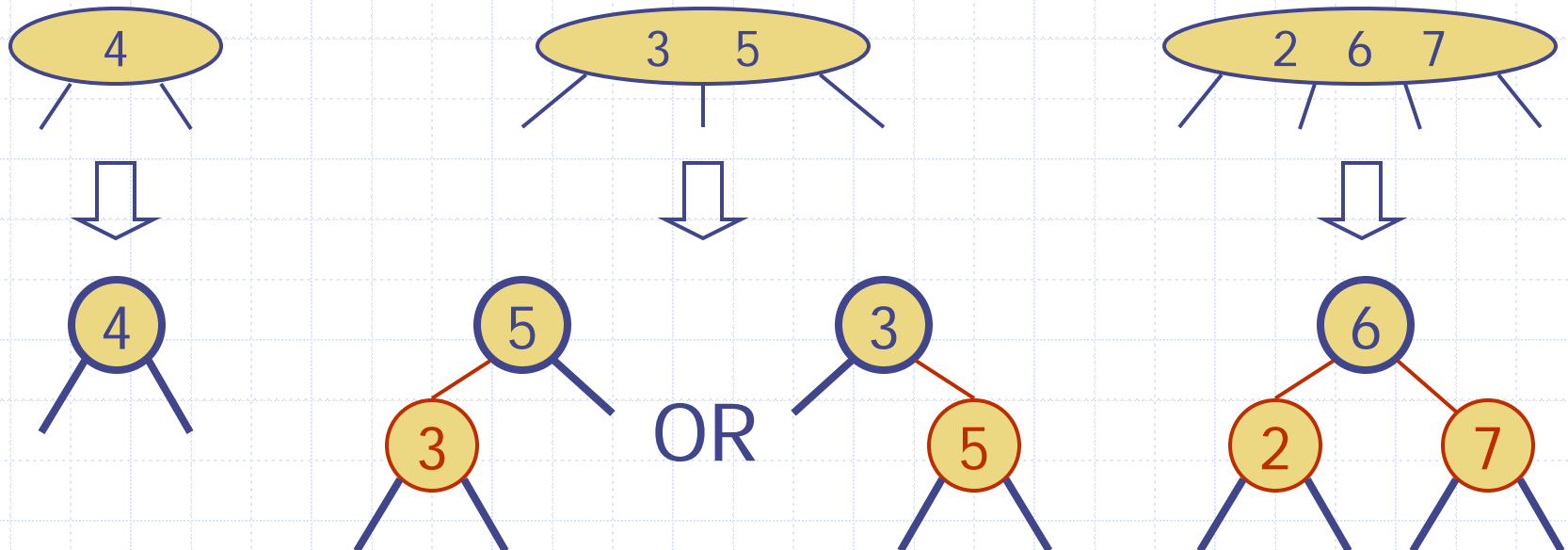


# Red-Black Trees



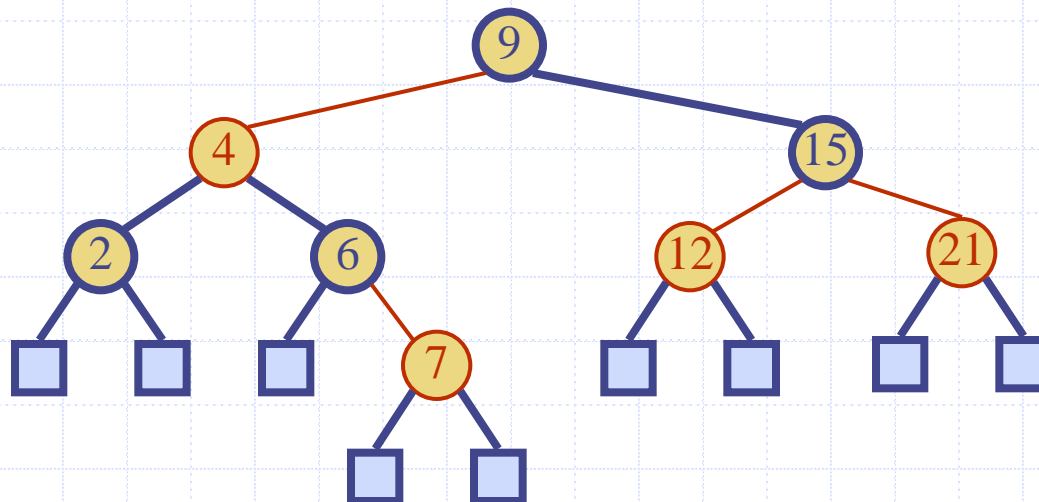
# From (2,4) to Red-Black Trees

- ◆ A red-black tree is a representation of a (2,4) tree by means of a binary tree whose nodes are colored **red** or **black**
- ◆ In comparison with its associated (2,4) tree, a red-black tree has
  - same logarithmic time performance
  - simpler implementation with a single node type



# Red-Black Trees

- ◆ A red-black tree can also be defined as a binary search tree that satisfies the following properties:
  - **Root Property:** the root is black
  - **External Property:** every leaf is black
  - **Internal Property:** the children of a red node are black
  - **Depth Property:** all the leaves have the same black depth



# Height of a Red-Black Tree

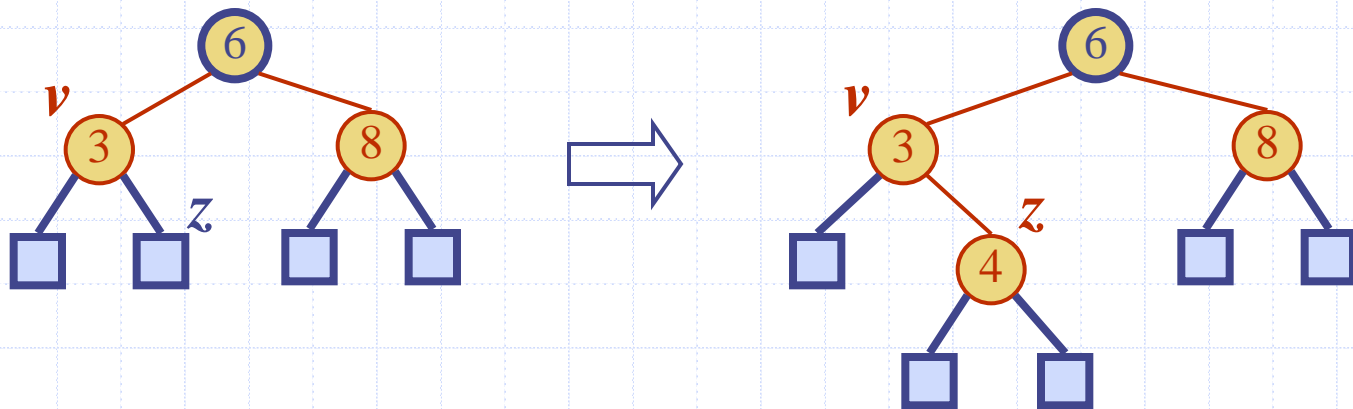
◆ **Theorem:** A red-black tree storing  $n$  entries has height  $O(\log n)$

Proof:

- The height of a red-black tree is at most twice the height of its associated (2,4) tree, which is  $O(\log n)$
- ◆ The search algorithm for a binary search tree is the same as that for a binary search tree
- ◆ By the above theorem, searching in a red-black tree takes  $O(\log n)$  time

# Insertion

- ◆ To perform operation  $\text{put}(k, o)$ , we execute the insertion algorithm for binary search trees and color **red** the newly inserted node  $z$  unless it is the root
  - We preserve the root, external, and depth properties
  - If the parent  $v$  of  $z$  is black, we also preserve the internal property and we are done
  - Else ( $v$  is red ) we have a **double red** (i.e., a violation of the internal property), which requires a reorganization of the tree
- ◆ Example where the insertion of 4 causes a double red:

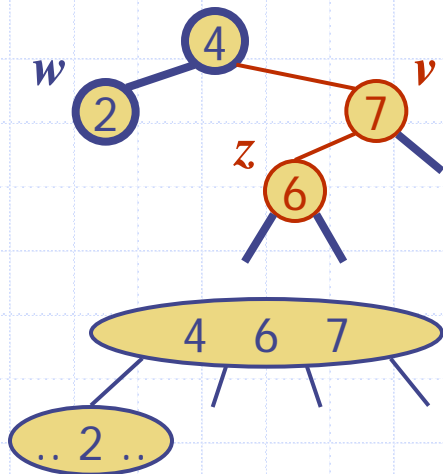


# Remedying a Double Red

- ◆ Consider a double red with child  $z$  and parent  $v$ , and let  $w$  be the sibling of  $v$

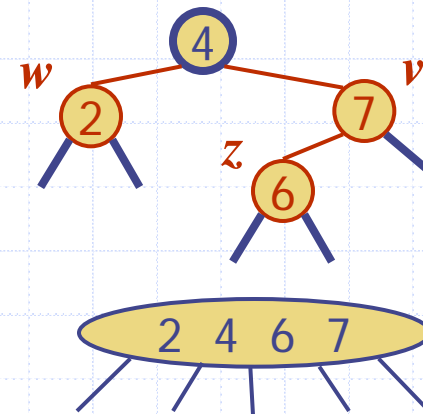
## Case 1: $w$ is black

- The double red is an incorrect replacement of a 4-node
- **Restructuring**: we change the 4-node replacement



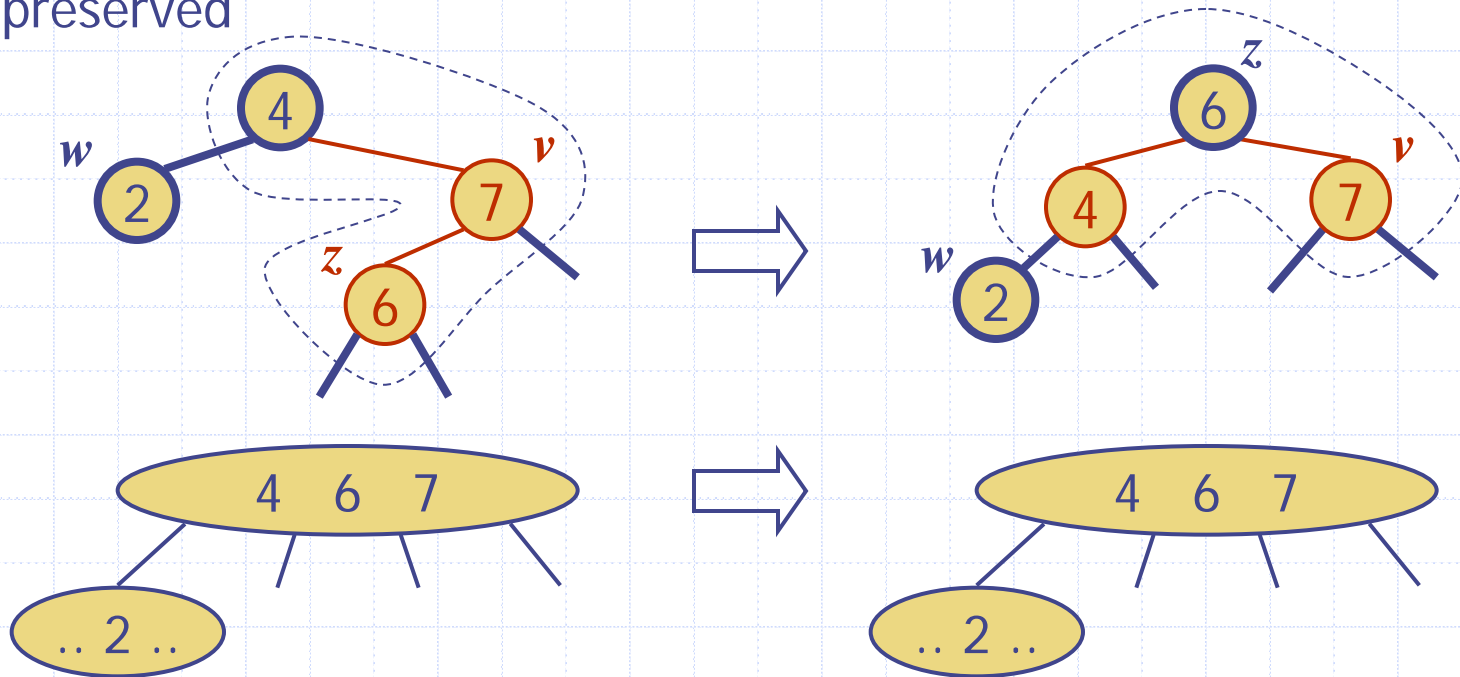
## Case 2: $w$ is red

- The double red corresponds to an overflow
- **Recoloring**: we perform the equivalent of a **split**



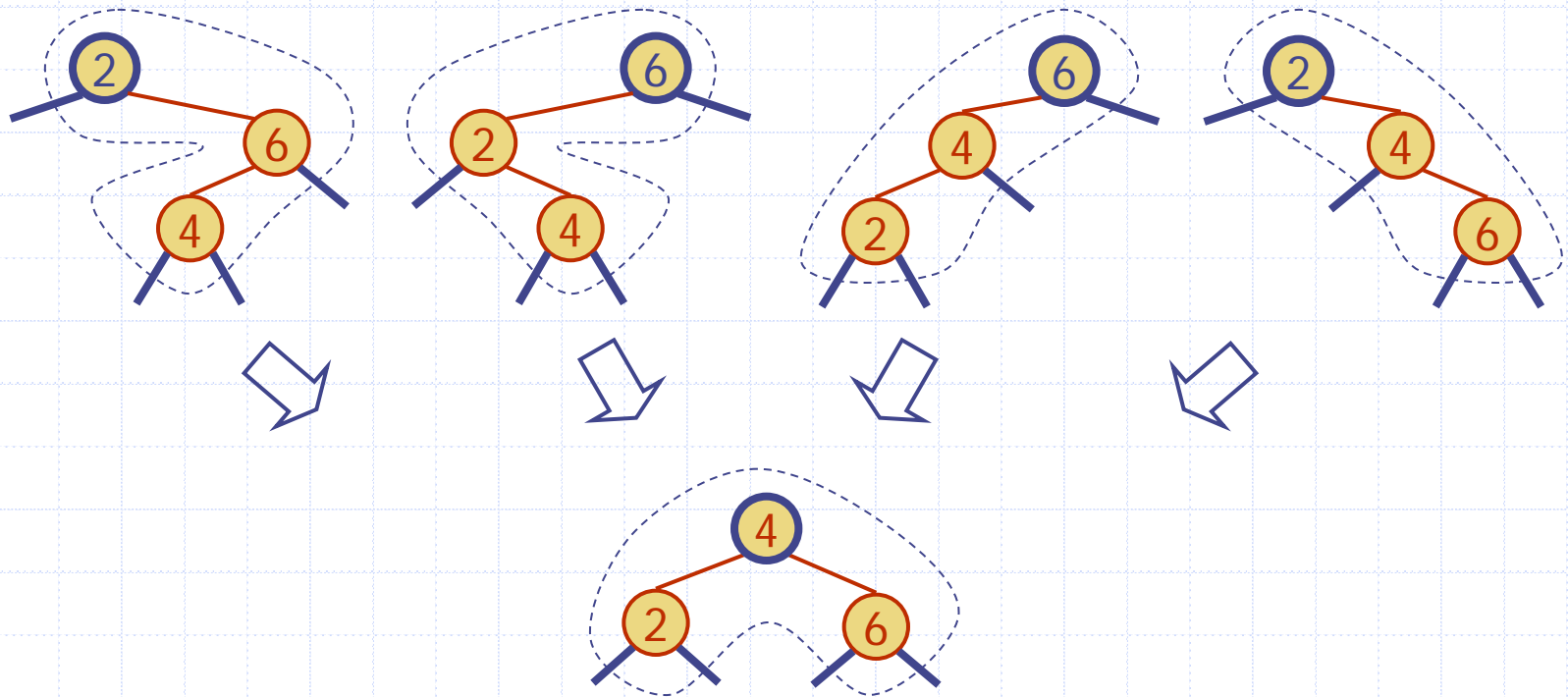
# Restructuring

- ◆ A restructuring remedies a child-parent double red when the parent red node has a black sibling
- ◆ It is equivalent to restoring the correct replacement of a 4-node
- ◆ The internal property is restored and the other properties are preserved



# Restructuring (cont.)

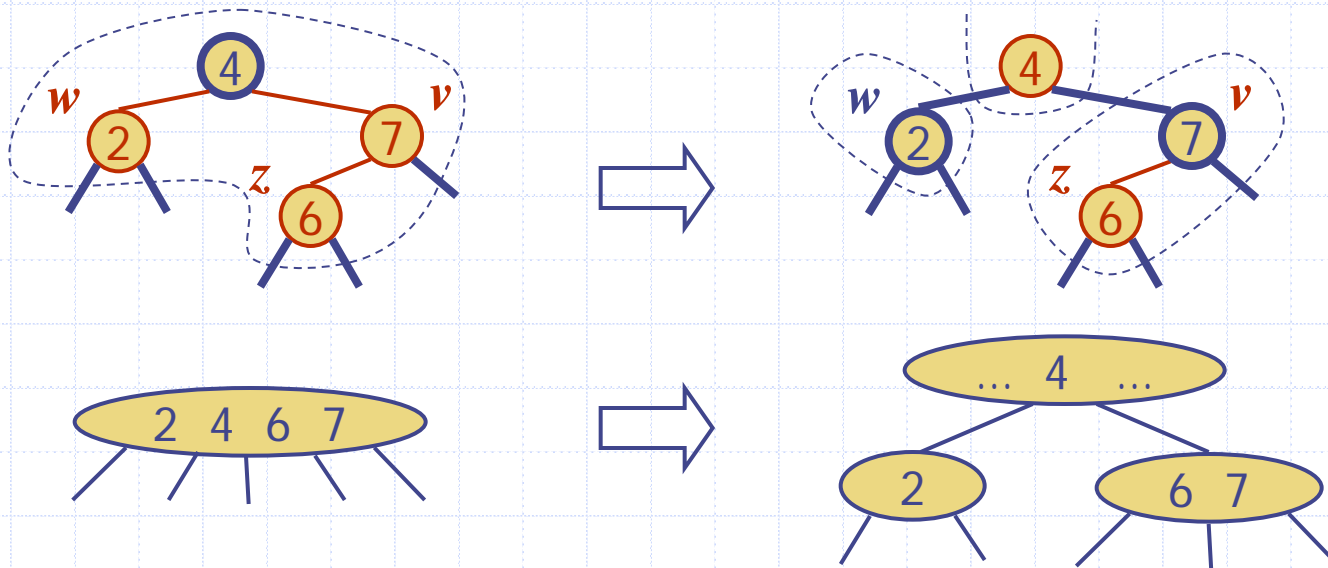
- ◆ There are four restructuring configurations depending on whether the double red nodes are left or right children





# Recoloring

- ◆ A recoloring remedies a child-parent double red when the parent red node has a red sibling
- ◆ The parent  $v$  and its sibling  $w$  become black and the grandparent  $u$  becomes red, unless it is the root
- ◆ It is equivalent to performing a split on a 5-node
- ◆ The double red violation may propagate to the grandparent  $u$



# Analysis of Insertion

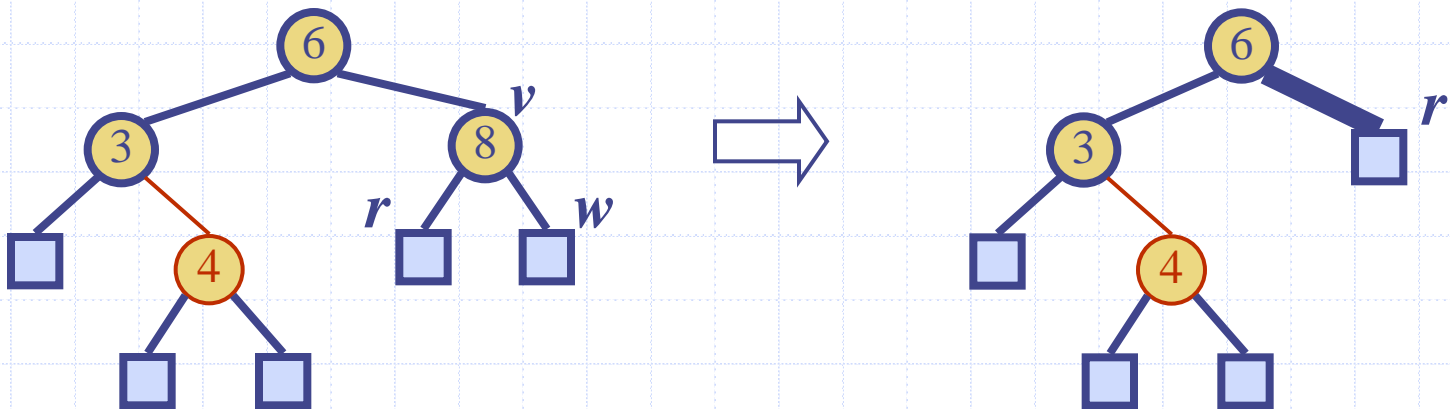
## Algorithm *put(k, o)*

1. We search for key  $k$  to locate the insertion node  $z$
2. We add the new entry  $(k, o)$  at node  $z$  and color  $z$  red
3. **while** *doubleRed*( $z$ )  
    **if** *isBlack*(*sibling*(*parent*( $z$ )))  
         $z \leftarrow \text{restructure}(z)$   
    **return**  
    **else** { *sibling*(*parent*( $z$ )) is red }  
         $z \leftarrow \text{recolor}(z)$

- ◆ Recall that a red-black tree has  $O(\log n)$  height
- ◆ Step 1 takes  $O(\log n)$  time because we visit  $O(\log n)$  nodes
- ◆ Step 2 takes  $O(1)$  time
- ◆ Step 3 takes  $O(\log n)$  time because we perform
  - $O(\log n)$  recolorings, each taking  $O(1)$  time, and
  - at most one restructuring taking  $O(1)$  time
- ◆ Thus, an insertion in a red-black tree takes  $O(\log n)$  time

# Deletion

- ◆ To perform operation **erase( $k$ )**, we first execute the deletion algorithm for binary search trees
- ◆ Let  $v$  be the internal node removed,  $w$  the external node removed, and  $r$  the sibling of  $w$ 
  - If either  $v$  or  $r$  was red, we color  $r$  black and we are done
  - Else ( $v$  and  $r$  were both black) we color  $r$  **double black**, which is a violation of the internal property requiring a reorganization of the tree
- ◆ Example where the deletion of 8 causes a double black:



# Remedying a Double Black

- ◆ The algorithm for remedying a double black node  $w$  with sibling  $y$  considers three cases

Case 1:  $y$  is black and has a red child

- We perform a **restructuring**, equivalent to a **transfer**, and we are done

Case 2:  $y$  is black and its children are both black

- We perform a **recoloring**, equivalent to a **fusion**, which may propagate up the double black violation

Case 3:  $y$  is red

- We perform an **adjustment**, equivalent to choosing a different representation of a 3-node, after which either Case 1 or Case 2 applies

- ◆ Deletion in a red-black tree takes  $O(\log n)$  time

# Red-Black Tree Reorganization

<b>Insertion</b> remedy double red		
Red-black tree action	(2,4) tree action	result
restructuring	change of 4-node representation	double red removed
recoloring	split	double red removed or propagated up

<b>Deletion</b> remedy double black		
Red-black tree action	(2,4) tree action	result
restructuring	transfer	double black removed
recoloring	fusion	double black removed or propagated up
adjustment	change of 3-node representation	restructuring or recoloring follows